Stability of plane-wave solutions to a dissipative generalization of the NLS equation

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July 21, 2008

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Outline

1. Review of NLS results
2. A dissipative NLS equation
3. Stability of plane-wave solutions to DNLS
4. Comments on vector DNLS

NLS

The (1+1)-dimensional cubic nonlinear Schrödinger equation is

$$i\psi_t + \alpha\psi_{xx} + \gamma|\psi|^2\psi = 0$$

- $\psi = \psi(x, t)$ is a complex-valued function
- α and γ are real constants

The class of plane-wave solutions is given by

$$\psi(x, t) = \psi_0 e^{ikx + i(-\alpha k^2 + \gamma\psi_0^2)t + i\xi}$$

- ψ_0, k and ξ are real constants

*These solutions have constant magnitude.

NLS Stability

Consider perturbed plane-wave solutions of the form

$$\psi_p(x,t) = \big(\psi_0 + \epsilon u(x,t) + i\epsilon v(x,t)\big)e^{ikx + i(-\alpha k^2 + \gamma\psi_0^2)t + i\xi}$$

where

- ϵ is a small real parameter
- $u(x,t)$ and $v(x,t)$ are real-valued functions

NLS Stability

WOLOG assume

$$u(x,t) = \overline{U}e^{ipx+\Omega t} + c.c.$$

$$v(x,t) = \overline{V}e^{ipx+\Omega t} + c.c.$$

where

- $\overline{U}$, $\overline{V}$ and Ω are complex constants
- p is a real constant
- $c.c.$ denotes complex conjugate

NLS Stability

This leads to the following system of algebraic equations

$$\begin{pmatrix} 2\gamma\psi_0^2 - \alpha p^2 & -\Omega - 2i\alpha kp \\ \Omega + 2i\alpha kp & -\alpha p^2 \end{pmatrix} \begin{pmatrix} \overline{U} \\ \overline{V} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

In order for this system to have a nonzero solution, Ω must be chosen to satisfy

$$\Omega_{\pm} = -2i\alpha kp \pm \sqrt{\alpha p^2(2\gamma\psi_0^2 - \alpha p^2)}$$

NLS Stability Observations

$$\Omega_{\pm} = -2i\alpha kp \pm \sqrt{\alpha p^2(2\gamma\psi_0^2 - \alpha p^2)}$$

- If $\alpha\gamma \leq 0$, then there is no instability.
- If $\alpha\gamma > 0$, then there is instability.
- The maximum growth rate is $|\gamma|\psi_0^2$.
- Note: k only contributes to the imaginary part of Ω.

DNLS

A dissipative generalization of NLS (DNLS) is

$$i\psi_t + (\alpha - ia)\psi_{xx} + (\gamma + ic)|\psi|^2\psi + id\psi = 0$$

- $\psi = \psi(x, t)$ is a complex-valued function
- α and γ are real constants
- a, c and d are nonnegative real constants

DNLS: A model of waves on deep water

The DNLS equation has arisen in many studies of water waves with dissipation

- Davey (1972) provides an argument for DNLS-like terms
- Lake *et al.* (1977) add the $id\psi$ term
- Blennerhassett (1980) derives an equation similar to DNLS
- Segur *et al.* (2005) add the $id\psi$ term
- Bridges & Dias (2007) study DNLS in a different manner

DNLS

Unlike NLS, DNLS does not (in general) conserve the $\mathcal{L}_2$-norm.

Specifically, the $\mathcal{L}_2$-norm is nonincreasing in t and

$$\frac{d}{dt}\Big(\int_0^{L_x} |\psi|^2 dx\Big) = -2\int_0^{L_x}\Big(a|\psi_x|^2 + c|\psi|^4 + d|\psi|^2\Big)dx$$

Note that if $a > 0$, then solutions with spatial dependence decay more rapidly than spatially-independent solutions.

DNLS Plane-Wave Solutions

DNLS admits four classes of solutions of the form

$$\psi(x,t) = \psi_0 \exp\left(ikx + \omega_r(t) + i\omega_i(t) + i\xi\right)$$

where

- ψ_0, k and ξ are real constants
- $\omega_r(t)$ and $\omega_i(t)$ are real-valued functions

DNLS Plane-Wave Solutions: Case #1

DNLS admits four classes of solutions of the form

$$\psi(x,t) = \psi_0 \exp\left(ikx + \omega_r(t) + i\omega_i(t) + i\xi\right)$$

where

- ψ_0, k and ξ are real constants
- $\omega_r(t)$ and $\omega_i(t)$ are real-valued functions

Case #1: $c = 0$ and $ak^2 + d = 0$

$$\omega_r(t) = 0,$$
$$\omega_i(t) = -t(\alpha k^2 - \gamma\psi_0^2).$$

*These solutions have constant magnitude.

DNLS Plane-Wave Solutions: Case #2

$$\psi(x,t) = \psi_0 \exp\left(ikx + \omega_r(t) + i\omega_i(t) + i\xi\right)$$

Case #2: $c > 0$ and $ak^2 + d = 0$

$$\omega_r(t) = -\frac{1}{2}\ln(1 + 2c\psi_0^2 t),$$
$$\omega_i(t) = -\alpha k^2 t - \frac{\gamma}{c}\omega_r(t).$$

*The magnitude of these solutions decays like $t^{-1/2}$.

DNLS Plane-Wave Solutions: Case #3

$$\psi(x,t) = \psi_0 \exp\left(ikx + \omega_r(t) + i\omega_i(t) + i\xi\right)$$

Case #3: $c = 0$ and $ak^2 + d > 0$

$$\omega_r(t) = -t(ak^2+d),$$
$$\omega_i(t) = -\alpha k^2 t + \frac{\gamma\psi_0^2}{2(ak^2+d)}\left(1 - e^{-2t(ak^2+d)}\right).$$

*The magnitude of these solutions decays exponentially.

DNLS Plane-Wave Solutions: Case #4

$$\psi(x,t) = \psi_0 \exp\big(ikx + \omega_r(t) + i\omega_i(t) + i\xi\big)$$

Case #4: $c > 0$ and $ak^2 + d > 0$

$$\omega_r(t) = \frac{1}{2}\ln\Big(\frac{ak^2+d}{(ak^2+d+c\psi_0^2)e^{2t(ak^2+d)} - c\psi_0^2}\Big),$$

$$\omega_i(t) = -t\big(\alpha k^2 + \frac{\gamma}{c}(ak+d)\big) - \frac{\gamma}{c}\omega_r(t).$$

*The magnitude of these solutions decays nearly exponentially.

DNLS Plane-Wave Solutions

These four cases include all plane-wave solutions of DNLS.

DNLS Stability

Consider perturbed solutions of the form

$$\psi_p(x,t) = \big(\psi_0 + \epsilon u(x,t) + i\epsilon v(x,t) + \mathcal{O}(\epsilon^2)\big)\exp(ikx + \omega_r(t) + i\omega_i(t) + i\xi)$$

where

- ϵ is a small real parameter
- $u(x,t)$ and $v(x,t)$ are real-valued functions

Note that any decay due to $\omega_r(t)$ has been factored out.

DNLS Stability Case #1 $c = 0$ and $ak^2 + d = 0$

Similar to the NLS result

$$\Omega_{\pm} = -ap^2 - 2i\alpha kp \pm \sqrt{\alpha p^2 (2\gamma\psi_0^2 - \alpha p^2)}$$

- $\Re(\Omega_{\pm})$ does not depend on k.
- The maximal growth rate is less than or equal to the corresponding NLS maximal growth rate.

DNLS Stability Case #2 $c > 0$ and $ak^2 + d = 0$

DNLS Stability Case #2 $c > 0$ and $ak^2 + d = 0$

WOLOG Assume

$$u(x,t) = U(t)e^{ipx} + c.c.$$
$$v(x,t) = V(t)e^{ipx} + c.c.$$

where

- $U(t)$, $V(t)$ are complex-valued functions
- p are real constants

DNLS Stability Case #2 $c > 0$ and $ak^2 + d = 0$

This leads to

$$\begin{pmatrix} U \\ V \end{pmatrix}' = \mathbf{A} \begin{pmatrix} U \\ V \end{pmatrix} + \mathbf{B}(t) \begin{pmatrix} U \\ V \end{pmatrix}$$

where

$$\mathbf{A} = \begin{pmatrix} -ap^2 - 2i\alpha kp & \alpha p^2 \\ -\alpha p^2 & -ap^2 - 2i\alpha kp \end{pmatrix}$$

$$\mathbf{B}(t) = \frac{2\psi_0^2}{1 + 2c\psi_0^2 t} \begin{pmatrix} -c & 0 \\ \gamma & 0 \end{pmatrix}$$

This system has a solution in terms of hypergeometric functions. These solutions are bounded for all t for all parameter choices.

DNLS Stability Cases #3, #4 $c \geq 0$ and $ak^2 + d > 0$

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Similar calculations lead to

$$\begin{pmatrix} U \\ V \end{pmatrix}' = \mathbf{A} \begin{pmatrix} U \\ V \end{pmatrix} + \mathbf{B}(t) \begin{pmatrix} U \\ V \end{pmatrix}$$

where

$$\mathbf{A} = \begin{pmatrix} -ap^2 - 2i\alpha kp & \alpha p^2 - 2iakp \\ -\alpha p^2 + 2iakp & -ap^2 - 2i\alpha kp) \end{pmatrix}$$

$$\mathbf{B}(t) = 2\psi_0^2 \mathrm{e}^{-2t(ak^2+d)} \begin{pmatrix} -c & 0 \\ \gamma & 0 \end{pmatrix}$$

Since $\mathbf{B}(t)$ decays exponentially, the stability of solutions of these forms is determined by the eigenvalues of $\mathbf{A}$.

DNLS Stability Cases #3, #4 $c \geq 0$ and $ak^2 + d > 0$

The eigenvalues of **A** are

$$\lambda_1 = -ap(2k+p) - i\alpha p(2k+p)$$

$$\lambda_2 = ap(2k-p) - i\alpha p(2k-p)$$

- If $a = 0$, then both λ_1 and λ_2 have zero real part. Thus, all corresponding solutions are neutrally stable.
- If $k = 0$, then both λ_1 and λ_2 have nonpositive real part. Thus, all corresponding solutions are linearly stable.
- If $a \neq 0$ and $k \neq 0$, then any choice of p for which

  $$p^2 < -2kp \quad \text{or} \quad p^2 < 2kp$$

  leads to an eigenvalue with positive real part. This establishes that all spatially-dependent solutions in this case are linearly unstable if a is positive.

DNLS Stability Cases #3, #4 $c \geq 0$ and $ak^2 + d > 0$

The maximum growth rate is

$$\max_p \Re(\lambda_1, \lambda_2) = ak^2$$

This growth rate is achieved if $p = -k$.

This establishes that the most unstable mode is the $p = -k$-mode.

DNLS Stability Cases #3, #4 $c \geq 0$ and $ak^2 + d > 0$

Recall

$$\psi_p(x,t) = \big(\psi_0 + \epsilon u(x,t) + i\epsilon v(x,t) + \mathcal{O}(\epsilon^2)\big)\exp(ikx + \omega_r(t) + i\omega_i(t) + i\xi)$$

The linear theory picks out the mode with the slowest overall decay rate (the 0-mode).

Growth versus decay

If $c > 0$, then the plane-wave solutions should decay to zero because the $\mathcal{L}_2$-norm is not conserved.

However, the calculations establish

- The maximum growth rate of an instability is ak^2.
- The decay rate due to $\omega_r(t)$ is $-(ak^2 + d)$.

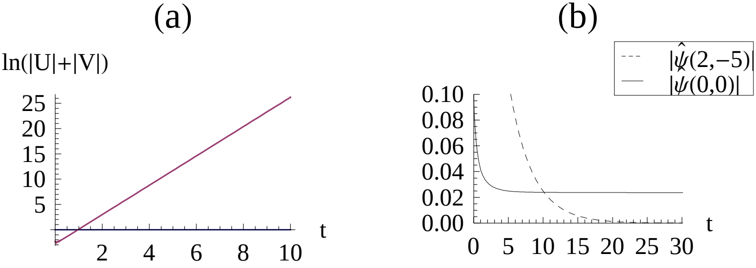
If $d = 0$, then the linear theory states that spatially-independent solutions do not decay if $c > 0$. Clearly, this is contrary to the fact that the $\mathcal{L}_2$-norm is not conserved when $c > 0$...........

Numerics help sort out the apparent contradiction.

DNLS Stability Cases #3, #4 Numerics

Consider the initial condition (a perturbed plane-wave solution)

$$\psi(x,y,0) = (1 + 0.1e^{-2ix+5iy})e^{2ix-5iy} = e^{2ix-5iy} + 0.1$$



(a) Results from numerical simulations of the linear ODEs.

(b) Results from numerical simulations of DNLS.

Summary

- All spatially-independent plane-wave solutions of DNLS are linearly stable.
- If $a = 0$, then all plane-wave solutions of DNLS are linearly stable.
- If $a > 0$, then all spatially-dependent plane-wave solutions of DNLS are linearly unstable.
- All of these results generalize to two and three dimensions.
- Work in progress: Generalizing these results to the vector DNLS equation.