Stability of nontrivial-phase solutions of higher-order generalizations of the cubic nonlinear Schrödinger equation

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Surface water waves

Consider an inviscid, incompressible, homogeneous fluid with a free surface resting on an impenetrable flat bed.

Let

- $\phi(\xi, \eta, \nu, \tau)$ denote the velocity potential
- $\zeta(\xi, \eta, \tau)$ denote the surface displacement
- g denote the acceleration due to gravity
- ξ and η denote horizontal spatial coordinates
- ν denote the vertical spatial coordinate
- τ denote time

Euler equations

If gravity is the only external force acting on the fluid, then the governing equations are

$$\phi_\nu = \zeta_\tau + \phi_\xi\zeta_\xi + \phi_\eta\zeta_\eta \qquad \text{at} \quad \nu = \zeta(\xi,\eta,\tau),$$

$$\phi_\tau + \frac{1}{2}|\nabla\phi|^2 + g\zeta = 0 \qquad \text{at} \quad \nu = \zeta(\xi,\eta,\tau),$$

$$\nabla^2\phi = 0 \qquad \text{for} \qquad -h < \nu < \zeta(\xi,\eta,\tau),$$

$$\phi_\nu = 0 \qquad \text{at} \qquad \nu = -h,$$

NLS

Let $\boldsymbol{\kappa} = (k, l)$ and a represent the characteristic wave number and characteristic amplitude of a packet of nearly monochromatic waves of small amplitude respectively. Let δk represent a small perturbation in k, let $\kappa^2 = k^2 + l^2$ and let ϵ be a small positive real number such that $\epsilon \ll 1$. Assume

1. small amplitude waves, $\kappa a = \mathcal{O}(\epsilon)$
2. nearly one-dimensional waves, $\frac{|l|}{k} = \mathcal{O}(\epsilon)$
3. slowly varying modulations, $\frac{\delta k}{k} = \mathcal{O}(\epsilon)$
4. deep water, $h = \infty$

The full equations through $\mathcal{O}(\epsilon^3)$ can be approximated by NLS

$$\boxed{i\psi_t + \frac{i}{2}\psi_x - \frac{1}{8}\psi_{xx} + \frac{1}{4}\psi_{yy} - \frac{1}{2}|\psi|^2\psi = 0}$$

MNLS

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Through $\mathcal{O}(\epsilon^4)$ gives MNLS (Dysthe, 1979)

$$i\psi_t + \frac{i}{2}\psi_x - \frac{1}{8}\psi_{xx} + \frac{1}{4}\psi_{yy} - \frac{1}{2}|\psi|^2\psi + \frac{3i}{8}\psi_{xyy} - \frac{i}{16}\psi_{xxx} + \frac{3i}{2}|\psi|^2\psi_x - \frac{i}{4}\psi^2\psi_x^* - \psi\bar{\phi}_x = 0$$

$$\bar{\phi}_z = \frac{1}{2}(|\psi|^2)_x \quad \text{at} \quad z = 0$$

$$\bar{\phi}_{xx} + \bar{\phi}_{yy} + \bar{\phi}_{zz} = 0 \quad \text{for} \quad -\infty < z < 0$$

$$\bar{\phi}_z = 0 \quad \text{at} \quad z = -\infty$$

NLS

A more general form of NLS is

$$i\psi_t + \alpha\psi_{xx} + \beta\psi_{yy} + \gamma|\psi|^2\psi = 0$$

Trivial-phase solutions of NLS

One-dimensional, trivial-phase (TP) solutions of NLS have the form

$$\psi(x,t) = \phi(x)e^{i\lambda t}$$

where

- λ is a real constant
- ϕ is a real-valued function

If $\alpha\gamma > 0$ (focusing case),

$$\phi(x) = \pm\sqrt{2\frac{\alpha}{\gamma}}\, bk\, \mathrm{cn}(bx,k), \qquad \lambda = \alpha b^2(2k^2-1)$$

$$\phi(x) = \pm\sqrt{2\frac{\alpha}{\gamma}}\, b\, \mathrm{dn}(bx,k), \qquad \lambda = \alpha b^2(2-k^2)$$

If $\alpha\gamma < 0$ (defocusing case),

$$\phi(x) = \pm\sqrt{-2\frac{\alpha}{\gamma}}\, bk\, \mathrm{sn}(bx,k), \quad \lambda = -\alpha b^2(1+k^2)$$

where b and k are free parameters.

Stability of TP solutions of NLS

The stability of TP solutions is well known.

In one dimension

- All *cn*-type solutions are unstable.
- All *dn*-type solutions are unstable.
- All *sn*-type solutions are stable.

In two dimensions

- All *cn*-type solutions are unstable.
- All *dn*-type solutions are unstable.
- All *sn*-type solutions are unstable.

Nontrivial-phase solutions of NLS

Nontrivial-phase solutions of NLS

One-dimensional, nontrivial-phase solutions (NTP) are of the form

$$\psi(x,t) = \phi(x)e^{i\theta(x)+i\lambda t}$$

where

- λ is a real constant
- ϕ and θ are real-valued functions

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$$\psi(x,t) = \phi(x)e^{i\theta(x)+i\lambda t}$$

where

$$\phi^2(x) = \frac{\alpha}{\gamma} b^2 \big(A\,\mathrm{sn}^2(bx,k) + B\big)$$

$$\theta(x) = c\int_0^x \phi^{-2}(\tau)d\tau$$

$$A = -2k^2$$

$$\lambda = \frac{1}{2}\alpha b^2(3B - 2(1+k^2))$$

$$c^2 = -\frac{\alpha^2 b^6}{2\gamma^2} B(B-2k^2)(B-2)$$

where b, k, B are free parameters.

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periodic: $\quad c = \dfrac{2\pi N}{\int_0^{4K/b}\phi^{-2}(\tau)d\tau}$

where b, k, B are free parameters and N is an integer.

Special limits: Focusing case

As $c \to 0$, NTP solutions limit to TP solutions.

If $\alpha\gamma > 0$ (focusing case),

- If $B \to (2k^2)^+$ then $c \to 0$ and

$$\phi^2(x) \to 2\frac{\alpha}{\gamma}\, b^2 k^2 \,\mathrm{cn}^2(bx, k)$$
$$\theta(x) \to 0$$

- If $B \to 2^-$ then $c \to 0$ and

$$\phi^2(x) \to 2\frac{\alpha}{\gamma}\, b^2 \,\mathrm{dn}^2(bx, k)$$
$$\theta(x) \to 0$$

Special limits: Defocusing case

As $c \to 0$, NTP solutions limit to TP solutions.

If $\alpha\gamma < 0$ (defocusing case),

- If $B \to 0^-$ then $c \to 0$ and

$$\phi^2(x) \to -2\frac{\alpha}{\gamma} b^2 k^2 \,\mathrm{sn}^2(bx, k)$$
$$\theta(x) \to 0$$

Stability of NTP solutions of NLS

Much less work has been done.

- In one dimension, all NTP solutions of focusing NLS are unstable.
- In one dimension, all NTP solutions of defocusing NLS are stable.
- In two dimensions, all NTP solutions are unstable.

Stability of NTP solutions of NLS

Some asymptotic results

- If $\alpha\beta < 0$, then all NTP solutions are unstable with respect to 2D perturbations with short wavelengths.
- If $\alpha\beta > 0$, then all NTP solutions are stable with respect to 2D perturbations with short wavelengths.
- Regardless of the sign of $\alpha\beta$, all NTP solutions are unstable with respect to 2D perturbations with long wavelengths in the transverse dimension.

A step backwards

Why are we interested in the stability of NTP solutions to MNLS

- Plane-wave solutions of NLS are unstable.
- Plane-wave solutions of MNLS are unstable.
- MNLS shows better agreement with some physical experiments.
- The growth rate of MNLS plane-wave instabilities can be higher than NLS plane-wave instabilities.

Are MNLS NTP solutions more or less unstable than NLS NTP solutions?

TP solutions of MNLS

There are none (other than the plane-wave solutions).

NTP solutions of MNLS

None are known analytically due to the nonlocal, nonlinear nature of the MNLS equation.

However, solutions can be found numerically.

Stability of NTP solutions of MNLS

Consider perturbed solutions with the following structure

$$\psi_p = (\phi(x) + \epsilon u(x, y, t) + i\epsilon v(x, y, t) + \mathcal{O}(\epsilon^2))e^{i\theta(x)+i\lambda t}$$

where

- ϵ is a small real parameter
- u and v are real-valued functions

NTP stability

Substituting ψ_p into MNLS, linearizing and separating into real and imaginary parts.
Without loss of generality, assume

$$u(x,y,t) = U(x,\rho)e^{i\rho y+\Omega t} + c.c.$$

$$v(x,y,t) = V(x,\rho)e^{i\rho y+\Omega t} + c.c.$$

where

- ρ is a real constant
- Ω is a complex constant
- U and V are complex-valued functions
- $c.c.$ denotes complex conjugate

NTP stability

This leads to a linear eigenvalue system.

In order to establish instability, one must find a bounded solution that corresponds to an Ω with positive real part.

NTP stability

- We have found numerical solutions of MNLS ignoring the nonlocal term.
- We have solved the linear eigenvalue problem (*) numerically using the Fourier-Floquet Hill method (SpectrUW).
- Preliminary results suggest that the maximum MNLS growth rate is larger than the maximum NLS growth rate.

Much work remains.......