Periodic Solutions of the Serre Equations

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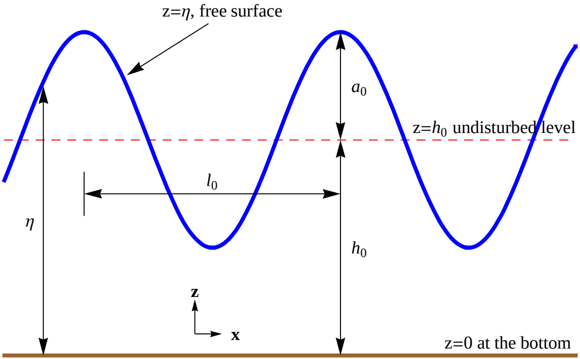
Outline

I. Physical system and governing equations

II. The Serre equations
   A. Derivation
   B. Justification
   C. Properties
   D. Solutions
   E. Stability

Physical System

Consider the 1-D flow of an inviscid, irrotational, incompressible fluid.



Governing Equations

Consider the 1-D flow of an inviscid, irrotational, incompressible fluid.

Let

- $\eta(x, t)$ represent the location of the free surface
- $u(x, z, t)$ represent the horizontal velocity of the fluid
- $w(x, z, t)$ represent the vertical velocity of the fluid
- $p(x, z, t)$ represent the pressure in the fluid

- $\epsilon = a_0/h_0$ (a measure of nonlinearity)
- $\delta = h_0/l_0$ (a measure of shallowness)
- $\Lambda = 2a_0/l_0 = 2\epsilon\delta$ (a measure of steepness)

Governing Equations

The dimensionless governing equations are

$$u_x + w_z = 0 \quad \text{for} \quad 0 < z < 1 + \epsilon\eta$$

$$u_z - \delta w_x = 0 \quad \text{for} \quad 0 < z < 1 + \epsilon\eta$$

$$\epsilon u_t + \epsilon^2 (u^2)_x + \epsilon^2 (uw)_z + p_x = 0 \quad \text{for} \quad 0 < z < 1 + \epsilon\eta$$

$$\delta^2 \epsilon w_t + \delta^2 \epsilon^2 u w_x + \delta^2 \epsilon^2 w w_z + p_z = -1 \quad \text{for} \quad 0 < z < 1 + \epsilon\eta$$

$$w = \eta_t + \epsilon u \eta_x \qquad \text{at} \qquad z = 1 + \epsilon\eta$$

$$p = 0 \qquad \text{at} \qquad z = 1 + \epsilon\eta$$

$$w = 0 \qquad \text{at} \qquad z = 0$$

Derivation of the Serre Equations

Depth Averaging

The Serre equations are obtained from the governing equations by (Serre 1953, Su & Gardner 1969, Green & Naghdi 1976)

1. Depth averaging

   The depth-averaged value of a quantity $f(x, z, t)$ is defined by

   $$\overline{f}(x, t) = \frac{1}{h(x, t)} \int_0^{h(x,t)} f(x, z, t)\, dz$$

   where $h(x, t) = 1 + \epsilon\eta(x, t)$ is the location of the free surface.

2. Assuming that $\delta << 1$
3. No restrictions are made on ϵ

Governing Equations

After depth averaging, the dimensionless governing equations are

$$\eta_t + \epsilon(\eta\overline{u})_x = 0$$

$$\overline{u}_t + \eta_x + \epsilon\overline{u}\,\overline{u}_x - \frac{\delta^2}{3\eta}\Big(\eta^3\big(\overline{u}_{xt} + \epsilon\overline{u}\,\overline{u}_{xx} - \epsilon(\overline{u}_x)^2\big)\Big)_x = \mathcal{O}(\delta^4, \epsilon\delta^4)$$

The Serre Equations

Truncating this system at $\mathcal{O}(\delta^4, \epsilon\delta^4)$ and transforming back to physical variables gives the Serre Equations

$$\eta_t + (\eta\overline{u})_x = 0$$

$$\overline{u}_t + g\eta_x + \overline{u}\,\overline{u}_x - \frac{1}{3\eta}\Big(\eta^3\big(\overline{u}_{xt} + \overline{u}\,\overline{u}_{xx} - (\overline{u}_x)^2\big)\Big)_x = 0$$

where

- $\eta(x,t)$ is the dimensional free-surface elevation
- $\overline{u}(x,t)$ is the dimensional depth-averaged horizontal velocity
- g is the acceleration due to gravity

Further justification of the Serre Equations

- Seabra-Santos *et al.* 1988: Compare range of validity of the Serre equations with other equations
- Dingemans 1997: Wave and current interactions
- Guizein & Barthélemy 2002: Soliton creation experiments
- Barthélemy 2003: Experiments of solitons over steps
- Cienfuegos *et al.* 2006: Serre equations for uneven bathymetries
- El & Grimshaw 2006: Serre equations modeling undular bores
- Lannes & Bonneton 2009: Serre equations are appropriate for nonlinear shallow water wave propagation

Serre equation conservation laws

Serre Equation Conservation Laws

I. Mass

$$\partial_t(\eta) + \partial_x(\eta\overline{u}) = 0$$

II. Momentum

$$\partial_t(\eta\overline{u}) + \partial_x\Big(\frac{1}{2}g\eta^2 - \frac{1}{3}\eta^3\overline{u}_{xt} + \eta\overline{u}^2 + \frac{1}{3}\eta^3\overline{u}_x^2 - \frac{1}{3}\eta^3\overline{u}\,\overline{u}_{xx}\Big) = 0$$

III. Energy

$$\partial_t\Big(\frac{1}{2}\eta\big(g\eta+\overline{u}^2+\frac{1}{3}\eta^2\overline{u}_x^2\big)\Big)+\partial_x\Big(\eta\overline{u}\big(g\eta+\frac{1}{2}\overline{u}^2+\frac{1}{2}\eta^2\overline{u}_x^2-\frac{1}{3}\eta^2(\overline{u}_{xt}+\overline{u}\overline{u}_{xx})\big)\Big) = 0$$

IV. Irrotationality

$$\partial_t\Big(\overline{u}-\eta\eta_x\overline{u}_x-\frac{1}{3}\eta^2\overline{u}_{xx}\Big)+\partial_x\Big(\eta\eta_t\overline{u}_x+g\eta-\frac{1}{3}\eta^2\overline{u}\,\overline{u}_{xx}+\frac{1}{2}\eta^2\overline{u}_x^2\Big) = 0$$

Translation Invariance

The Serre equations are invariant under the transformation

$$\eta(x,t) = \hat{\eta}(x - st, t)$$
$$\overline{u}(x,t) = \hat{u}(x - st, t) + s$$
$$\hat{x} = x - st$$

where s is any real parameter.

Physically, this corresponds to adding a constant horizontal flow to the entire system.

Solutions of the Serre Equations

Solutions of the Serre Equations

$$\eta(x,t) = a_0 + a_1 \mathrm{dn}^2\big(\kappa(x-ct), k\big)$$

$$\overline{u}(x,t) = c\Big(1 - \frac{h_0}{\eta(x,t)}\Big)$$

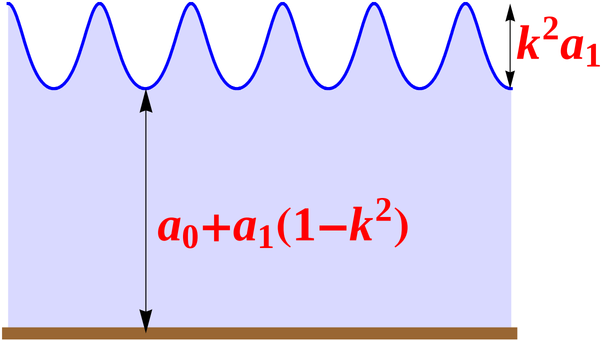
$$\kappa = \frac{\sqrt{3a_1}}{2\sqrt{a_0(a_0+a_1)(a_0+(1-k^2)a_1)}}$$

$$c = \frac{\sqrt{ga_0(a_0+a_1)(a_0+(1-k^2)a_1)}}{h_0}$$

$$h_0 = a_0 + a_1\frac{E(k)}{K(k)}$$

Here $k \in [0,1]$, $a_0 > 0$, and $a_1 > 0$ are free parameters.

The water surface if $k \in (0, 1)$:

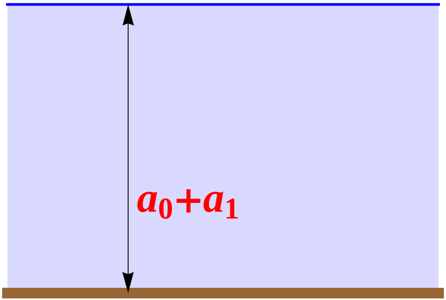


Constant Solution of the Serre Equations

If $k = 0$, the solution simplifies to

$$\eta(x,t) = a_0 + a_1$$
$$\overline{u}(x,t) = 0$$

The water surface if $k = 0$:



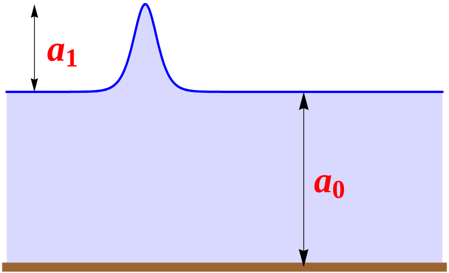
Soliton Solution of the Serre Equations

If $k = 1$, the solution reduces to

$$\eta(x,t) = a_0 + a_1 \operatorname{sech}^2\left(\frac{\sqrt{3a_1}}{2a_0\sqrt{a_0+a_1}}\left(x - \sqrt{g(a_0+a_1)}\,t\right)\right)$$

$$\overline{u}(x,t) = \sqrt{g(a_0+a_1)}\left(1 - \frac{a_0}{\eta(x,t)}\right)$$

The water surface if $k = 1$:



Stability of Periodic Solutions of the Serre Equations

The soliton solution of the Serre equations is linearly stable (Li 2001).

Stability of Solutions of the Serre Equations

Transform to a moving coordinate frame

$$\chi = x - ct$$

$$\tau = t$$

The Serre equations become

$$\eta_\tau - c\eta_\chi + \left(\eta\overline{u}\right)_\chi = 0$$

$$\overline{u}_\tau - c\overline{u}_\chi + \overline{u}\,\overline{u}_\chi + \eta_\chi - \frac{1}{3\eta}\Big(\eta^3\big(\overline{u}_{\chi\tau} - c\overline{u}_{\chi\chi} + \overline{u}\,\overline{u}_{\chi\chi} - (\overline{u}_\chi)^2\big)\Big)_\chi = 0$$

and the solutions become

$$\eta = \eta_0(\chi) = a_0 + a_1\mathrm{dn}^2(\kappa\chi, k)$$

$$\overline{u} = u_0(\chi) = c\Big(1 - \frac{h_0}{\eta_0(\chi)}\Big)$$

Stability of Solutions of the Serre Equations

Consider perturbed solutions of the form

$$\eta_{\text{pert}}(\chi,\tau) = \eta_0(\chi) + \epsilon\eta_1(\chi,\tau) + \mathcal{O}(\epsilon^2)$$
$$\overline{u}_{\text{pert}}(\chi,\tau) = u_0(\chi) + \epsilon u_1(\chi,\tau) + \mathcal{O}(\epsilon^2)$$

where

- ϵ is a small real parameter
- $\eta_1(\chi,\tau)$ and $u_1(\chi,\tau)$ are real-valued functions

- $\eta_0(\chi) = a_0 + a_1\text{dn}^2(\kappa\chi, k)$
- $u_0(\chi) = c\Big(1 - \frac{h_0}{\eta_0(\chi)}\Big)$

Stability of Solutions of the Serre Equations

Without loss of generality, assume

$$\eta_1(\chi,\tau) = H(\chi)e^{\Omega\tau} + c.c.$$
$$u_1(\chi,\tau) = U(\chi)e^{\Omega\tau} + c.c.$$

where

- $H(\chi)$ and $U(\chi)$ are complex-valued functions
- Ω is a complex constant
- $c.c.$ denotes complex conjugate

This leads the following linear system

$$\mathcal{L}\begin{pmatrix} H \\ U \end{pmatrix} = \Omega\,\mathcal{M}\begin{pmatrix} H \\ U \end{pmatrix}$$

where

$$\mathcal{L} = \begin{pmatrix} -u_0' + (c - u_0)\partial_\chi & -\eta_0' - \eta_0\partial_\chi \\ \mathcal{L}_{21} & \mathcal{L}_{22} \end{pmatrix}$$

$$\mathcal{M} = \begin{pmatrix} 1 & 0 \\ 0 & 1 - \eta_0\eta_0'\partial_\chi - \frac{1}{3}\eta_0^2\partial_{\chi\chi} \end{pmatrix}$$

and prime represents derivative with respect to χ.

where

$$\mathcal{L}_{21} = -\eta_0'(u_0')^2 - c\eta_0'u_0'' - \frac{2}{3}c\eta_0 u_0''' + \eta_0' u_0 u_0'' - \frac{2}{3}\eta_0 u_0' u_0'' + \frac{2}{3}\eta_0 u_0 u_0''' + \left(\eta_0 u_0 u_0'' - g - \eta_0(u_0')^2 - c\eta_0 u_0''\right)\partial_\chi$$

$$\mathcal{L}_{22} = -u_0' + \eta_0\eta_0'u_0'' + \frac{1}{3}\eta_0^2 u_0''' + \left(c - u_0 - 2\eta_0\eta_0'u_0' - \frac{1}{3}\eta_0^2 u_0''\right)\partial_\chi + \left(\eta_0\eta_0'u_0 - c\eta_0\eta_0' - \frac{1}{3}\eta_0^2 u_0'\right)\partial_{\chi\chi} + \left(\frac{1}{3}\eta_0^2 u_0 - \frac{1}{3}c\eta_0^2\right)\partial_{\chi\chi\chi}$$

$$\mathcal{L}\begin{pmatrix} H \\ U \end{pmatrix} = \Omega\,\mathcal{M}\begin{pmatrix} H \\ U \end{pmatrix}$$

We solved this system using the Fourier-Floquet-Hill Method (Deconinck & Kutz 2006).

This method allows the computation of eigenvalues corresponding to eigenfunctions of the form

$$\begin{pmatrix} H \\ U \end{pmatrix} = e^{i\rho\chi}\begin{pmatrix} H^P \\ U^P \end{pmatrix}$$

where

- H^P and U^P are periodic in χ with period $2K/\kappa$
- $\rho \in [-\pi\kappa/(4K), \pi\kappa/(4K))$.

If $\rho = 0$, then the perturbation has the same period as the unperturbed solution.

Stability of Solutions of the Serre Equations

We conducted three series of numerical simulations

I. Fixed a_0 and k, varying a_1
II. Fixed a_0 and a_1, varying k
III. Fixed a_0 and wave amplitude, varying k and a_1 simultaneously

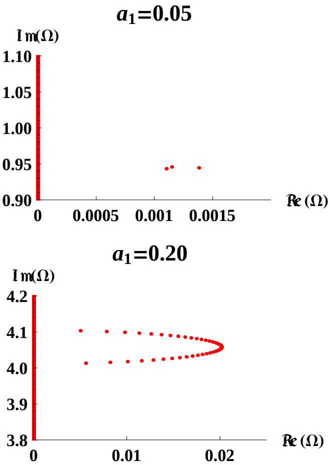
wave amplitude$=a_1 k^2$

Case I: Fixed a_0 and k

$$a_0 = 0.3,\ k = 0.75,\ N = 75,\ P = 1500$$

Case I: Fixed a_0 and k			
a_1	δ	ϵ	Λ
0.05	0.0922	0.0420	0.0077
0.1	0.1302	0.0762	0.0198
0.2	0.1841	0.1284	0.0473
0.3	0.2258	0.1664	0.0751

Case I: Fixed a_0 and k



Case I: Fixed a_0 and k

Observations:

- If a_1 is small enough, there are no Ωs with positive real part. Therefore the solution is linearly stable.
- If a_1 is large enough, then there are Ωs with positive real part. Therefore the solution is linearly unstable.
- The cutoff between stability and instability is at $a_1 \approx 0.023$.
- The maximum growth rate increases as a_1 increases.

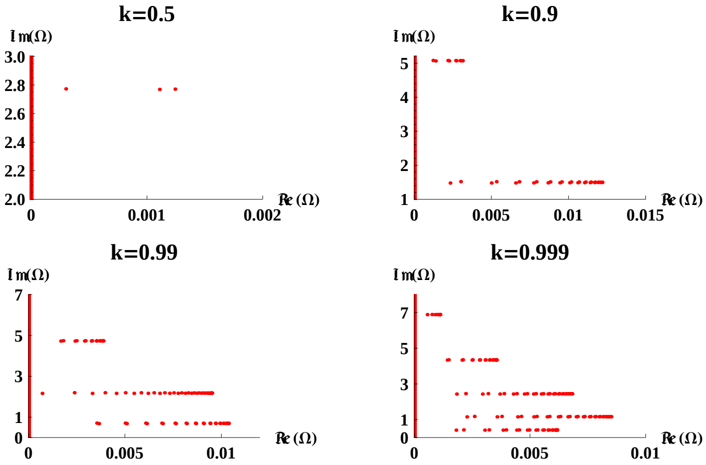
- All instabilities are oscillatory instabilities.
- The rate of instability oscillation increases as a_1 increases.
- For each value of a_1, there is only one band of instabilities (for these parameters).
- Generally, the instability with maximal growth rate corresponds to a perturbation with nonzero ρ.

Case II: Fixed a_0 and a_1

$$a_0 = 0.3,\ a_1 = 0.1,\ N = 75,\ P = 1500$$

Case II: Fixed a_0 and a_1			
k	δ	ϵ	Λ
0.5	0.1482	0.0323	0.0096
0.9	0.1078	0.1153	0.0249
0.99	0.0708	0.1482	0.0210
0.999	0.0517	0.1548	0.0160

Case II: Fixed a_0 and a_1



Case II: Fixed a_0 and a_1

Observations:

- If k is small enough, there is no instability.
- If k is large enough, there is instability.
- The cutoff between stability and instability occurs at $k \approx 0.30$.
- As k increases away from zero, the maximum growth rate increases until $k \approx 0.947$. Above this value, the maximum growth rate decreases. At $k = 0.947$, the maximum growth rate is 0.0132.

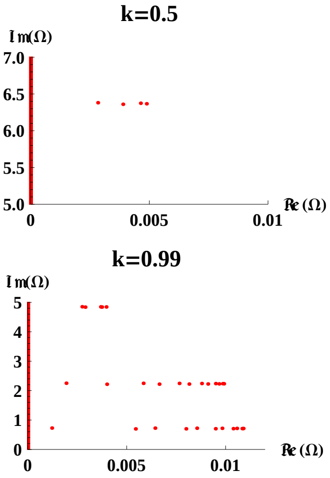
- All instabilities are oscillatory instabilities.
- As k increases, the number of bands of instability increases.
- Generally, the instability with maximal growth rate corresponds to a $\rho \neq 0$ perturbation.

Case III: Fixed a_0 and wave amplitude

$$a_0 = 0.3,\ a_1 k^2 = 0.1,\ N = 75,\ P = 1500$$

Case III: Fixed a_0 and wave amplitude			
k	δ	ϵ	Λ
0.5	0.2967	0.0771	0.0458
0.9	0.1196	0.1376	0.0329
0.99	0.0715	0.1509	0.0216
0.999	0.0518	0.1551	0.0161

Case III: Fixed a_0 and wave amplitude



Case III: Fixed a_0 and wave amplitude

Observations:

- If k is small enough, there is no instability.
- If k is large enough, there is instability.
- The cutoff between stability and instability occurs at $k \approx 0.102$.
- As k increases away from zero, the maximum growth rate increases until $k \approx 0.86$. Above this value, the maximum growth rate decreases. At $k = 0.86$, the maximum growth rate is 0.020.

- All instabilities are oscillatory.
- As k increases, the number of bands of instabilities increases.
- Generally, the instability with maximal growth rate corresponds to a $\rho \neq 0$ perturbation.

Summary

- Waves with sufficiently small amplitude/steepness are stable.
- Waves with sufficiently large amplitude/steepness are unstable.
- Series of simulations in which Λ was unbounded, did not exhibit a maximal growth rate.
- Series of simulations in which Λ was bounded, exhibited a maximal growth rate.